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NOTE

On the Coefficients and Zeros of a Polynomial

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Let $p(z)=1+\sum_{j=1}^{n} b_j z^j$ be a complex polynomial. Two theorems on the coefficients and zeros of $p(z)$ are proved in this paper.

Let

$$p(z)=1+b_1 z+\cdots+b_n z^n$$

be a complex polynomial. There is a close connection between the location of the zeros of $p(z)$ and its coefficients. Some results on the connection were given by Alzer [1] and Rahman [2].

The following result is a version of Alzer's Theorem.

THEOREM A. *Let $p(z)=1+b_1 z+\cdots+b_n z^n$ with $n\geqslant 2$ and $b_n\neq 0$. If $p(z_0)=0$, then*

$$|z_0|\leqslant\left|\frac{b_{n-1}}{b_n}\right|+\left(\sum_{j=2}^{n}\left|\frac{b_{n-j}}{b_n}\right|\alpha^{j-2}\right)^{1/2}, \tag{1}$$

where

$$\alpha=\left(\max_{2\leqslant j\leqslant n}\left|\frac{b_{n-j}}{b_n}\right|^{1/j}\right)^{-1}, \qquad b_0=1.$$

In this note, we prove two theorems on the coefficients and zeros of a polynomial.

THEOREM 1. *Suppose that $p(z)=1+b_1 z+\cdots+b_n z^n$ with $b_n\neq 0$. If $p(z_0)=0$, then*

$$\left(\sum_{j=1}^{n}|b_j|\,\beta^{j-1}\right)^{-1}\leqslant|z_0|\leqslant\sum_{j=1}^{n}\left|\frac{b_{n-j}}{b_n}\right|\gamma^{j-1}, \tag{2}$$

where $\beta=1/\max_{1\leqslant j\leqslant n}|b_j|^{1/j}$, and $\gamma=1/\max_{1\leqslant j\leqslant n}|b_{n-j}/b_n|^{1/j}$ $(b_0=1)$.

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THEOREM 2. *Suppose that* $p(z) = 1 + b_1 z + \cdots + b_n z^n$ *is nonvanishing in* $|z| < 1$. *Then*

(i) $$|b_n| \leqslant 1; \tag{3}$$

(ii) *for* $k = 1, 2, ..., n-1$ (*when* $n \geqslant 2$),

$$|b_k| \leqslant \binom{n-1}{k} + \binom{n-1}{k-1} |b_n| \tag{4}$$

(*with the convention* $\binom{n-1}{0} = 1$). *The estimate* (4) *is the best possible as the function* $p(z) = (1+z)^{n-1}(1 + |b_n| z)$ *shows.*

Proof of Theorem 1. Write $K = \sum_{j=1}^{n} |b_j| \beta^{j-1}$. Then

$$\beta K = \sum_{j=1}^{n} |b_j| \beta^j \geqslant 1.$$

Hence, for $j = 1, 2, ..., n$,

$$|b_j| \beta^{j-1} K^{j-1} \geqslant |b_j|.$$

Thus, for $|z| < 1/K$,

$$\left| \sum_{j=1}^{n} b_j z^j \right| < \sum_{j=1}^{n} |b_j| K^{-j} \leqslant \sum_{j=1}^{n} |b_j| \beta^{j-1} K^{-1} = 1.$$

Hence we obtain that, for $|z| < 1/K$,

$$|p(z)| \geqslant 1 - \left| \sum_{j=1}^{n} b_j z^j \right| > 0.$$

This implies that if $p(z_0) = 0$, then $|z_0| \geqslant 1/K$. The first inequality in (2) is proved.

Let

$$q(\zeta) = \frac{1}{b_n} \zeta^n p\left(\frac{1}{\zeta}\right) = 1 + \frac{b_{n-1}}{b_n} \zeta + \frac{b_{n-2}}{b_n} \zeta^2 + \cdots + \frac{b_1}{b_n} \zeta^{n-1} + \frac{1}{b_n} \zeta^n.$$

Since $p(z_0) = 0$ we have $q(1/z_0) = 0$. It follows from what we have just proved that

$$\left| \frac{1}{z_0} \right| \geqslant \frac{1}{\sum_{j=1}^{n} \left| \frac{b_{n-j}}{b_n} \right| \gamma^{j-1}},$$

where

$$\gamma = \frac{1}{\max_{1 \leqslant j \leqslant n} \left| \frac{b_{n-j}}{b_n} \right|^{1/j}} \qquad \text{and} \qquad b_0 = 1.$$

This proves the second inequality of (2).

We point out that, in some instances, the upper bound of zeros given by (2) is better than the one given by (1). For instance, let

$$p(z) = 1 + b_{n-1} z^{n-1} + b_n z^n$$

with $b_n \neq 0$ and $p(z_0) = 0$. Then by Theorem A we have

$$|z_0| \leqslant \left| \frac{b_{n-1}}{b_n} \right| + |b_n|^{-1/n}.$$

However, by Theorem 1 we obtain

$$|z_0| \leqslant \left| \frac{b_{n-1}}{b_n} \right| + \frac{\gamma^{n-1}}{|b_n|},$$

where $\gamma = 1/\max(|b_{n-1}/b_n|, |1/b_n|^{1/n}) \leqslant |b_n|^{1/n}$. We see that

$$\left| \frac{b_{n-1}}{b_n} \right| + \frac{\gamma^{n-1}}{|b_n|} \leqslant \left| \frac{b_{n-1}}{b_n} \right| + |b_n|^{-1/n}.$$

The proof of Theorem 2 needs the following lemmas.

LEMMA 1. *Let $x_1, x_2, ..., x_n$ be real numbers and $c = x_1 x_2 \cdots x_n$.*

(i) *If $0 \leqslant x_k \leqslant 1$ for $k = 1, 2, ..., n$, then*

$$x_1 + x_2 + \cdots + x_n \leqslant n - 1 + c. \tag{5}$$

(ii) *If $x_k \geqslant 1$ for $k = 1, 2, ..., n$, then (5) holds also.*

(iii) *If $0 < x_k \leqslant 1$ for $k = 1, 2, ..., n$, then*

$$\frac{c}{x_1} + \frac{c}{x_2} + \cdots + \frac{c}{x_n} \leqslant 1 + (n-1)\, c. \tag{6}$$

Proof. We prove (i) by mathematical induction. For $n=1$, (5) is trivial. Assume (5) holds for $n-1$. We can assume that $x_1 x_2 \cdots x_n \neq 0$. Hence $x_n \neq 0$ and $c \leqslant x_n \leqslant 1$. Thus

$$\begin{aligned} x_1+x_2+\cdots+x_{n-1}+x_n &\leqslant n-2+x_1 x_2 \cdots x_{n-1}+x_n \\ &= n-2+\frac{c}{x_n}+x_n \\ &\leqslant n-2+1+c \\ &= n-1+c. \end{aligned}$$

This proves (i).

Similarly, we can prove (ii).

If $0<x_k \leqslant 1$ for $k=1, 2, ..., n$, then by (ii)

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n} \leqslant n-1+\frac{1}{x_1 x_2 \cdots x_n}.$$

It follows that (6) holds. This completes the proof of Lemma 1.

Let $a_1, a_2, ..., a_n$ be real numbers. For $k=1, 2, ..., n$ let $(a_1, a_2, ..., a_n; k)$ denote the coefficient of x^k in the polynomial $\prod_{j=1}^{n}(1+a_j x)$. We have the following result.

LEMMA 2. *Let $x_1, x_2, ..., x_n$ be real numbers and $c=x_1 x_2 \cdots x_n$ with $n \geqslant 2$. If $0 \leqslant x_j \leqslant 1$ for $j=1, 2, ..., n$, then for $k=1, 2, ..., n-1$*

$$(x_1, x_2, ..., x_n; k) \leqslant \binom{n-1}{k}+\binom{n-1}{k-1} c \tag{7}$$

with the convention $\binom{n-1}{0}=1$.

Proof. For $n \geqslant 2$, part (i) of Lemma 1 shows that

$$(x_1, x_2, ..., x_n; 1)=x_1+x_2+\cdots+x_n \leqslant n-1+c=\binom{n-1}{1}+\binom{n-1}{0} c.$$

If $c \neq 0$, then $0<x_j \leqslant 1$, $j=1, 2, ..., n$. Part (iii) of Lemma 1 shows that

$$\begin{aligned} (x_1, x_2, ..., x_n; n-1) &= \frac{c}{x_1}+\frac{c}{x_2}+\cdots+\frac{c}{x_n} \leqslant 1+(n-1) c \\ &= \binom{n-1}{n-1}+\binom{n-1}{n-2} c. \end{aligned}$$

If $c=0$, then at least one x_j is zero. We can assume that $x_1=0$. Then

$$(x_1, x_2, ..., x_n; n-1) = x_2 x_3 \cdots x_n \leqslant 1 = \binom{n-1}{n-1} + \binom{n-1}{n-2} c.$$

Now we have proved that (7) holds for $k=1$ and $k=n-1$ whenever $n \geqslant 2$.

Now we proceed by mathematical induction. For $n=2$ and $n=3$ we have proved that the claim of Lemma 2 holds as the above shows. Assume it holds for n. For $n+1$ it is sufficient to prove that

$$(x_1, x_2, ..., x_n, x_{n+1}; k) \leqslant \binom{n}{k} + \binom{n}{k-1} x_1 x_2 \cdots x_n x_{n+1}$$

for $k=2, 3, ..., n-1$. We can assume that $x_1 \neq 0$. Then for $k=2, 3, ..., n-1$

$$\begin{aligned}
&(x_1, x_2, ..., x_n, x_{n+1}; k) \\
&\quad = x_1(x_2, x_3, ..., x_{n+1}; k-1) + (x_2, x_3, ..., x_{n+1}; k) \\
&\quad \leqslant x_1 \left[\binom{n-1}{k-1} + \binom{n-1}{k-2} x_2 x_3 \cdots x_{n+1} \right] + \binom{n-1}{k} \\
&\qquad + \binom{n-1}{k-1} x_2 x_3 \cdots x_{n+1} \\
&\quad = \binom{n-1}{k-1} \left(x_1 + \frac{x_1 x_2 \cdots x_{n+1}}{x_1} \right) + \binom{n-1}{k-2} x_1 x_2 \cdots x_{n+1} + \binom{n-1}{k} \\
&\quad \leqslant \binom{n-1}{k-1} (1 + x_1 x_2 + \cdots x_{n+1}) + \binom{n-1}{k-2} x_1 x_2 \cdots x_{n+1} + \binom{n-1}{k} \\
&\quad = \binom{n}{k} + \binom{n}{k-1} x_1 x_2 \cdots x_{n+1}.
\end{aligned}$$

Hence the claim holds for $n+1$ and this completes the proof of Lemma 2.

Proof of Theorem 2. Let $p(z) = 1 + b_1 z + \cdots + b_n z^n = \prod_{j=1}^{n} (1 + \alpha_j z)$. Since $p(z)$ is nonvanishing in $|z| < 1$, we have $|\alpha_j| \leqslant 1$ for $j=1, 2, ..., n$. Hence

$$|b_n| = |\alpha_1 \alpha_2 \cdots \alpha_n| \leqslant 1.$$

By Lemma 3, for $k=1, 2, ..., n-1$ (when $n \geqslant 2$) we have

$$|b_k| \leqslant (|\alpha_1|, |\alpha_2|, ..., |\alpha_n|; k) \leqslant \binom{n-1}{k} + \binom{n-1}{k-1} |b_n|.$$

This completes the proof of Theorem 2.

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